NOTES AND DISCUSSIONS

The rate of evolution of a quantum state

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How long does it take for an isolated quantum system to evolve to an orthogonal state? In a recent note in this journal,[1] Vaidman complained that it is not easy to find exact limits for this problem in the literature. It is the purpose of this note to supply some historical background and report some related results.

The earliest discussion of the above problem known to me is the classic but apparently little-read article by Mandelstam and Tamm of 1945.[2] In this paper, Mandelstam and Tamm prove the inequality,

$$|\langle\psi(0)|\psi(t)\rangle|^2 \geqslant \cos^2\left(\frac{\Delta H t}{\hbar}\right), \qquad \text{for } 0 \leqslant t \leqslant \frac{\pi\hbar}{2\Delta H}, \tag{1}$$

where $|\psi(0)\rangle$ is the state at $t=0$ and $\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$ is the uncertainty in energy. Note that ΔH is independent of time.

Mandelstam and Tamm derive this inequality in a very simple manner, starting from their better-known[3] relation,

$$\Delta H \Delta A \geqslant \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right|. \tag{2}$$

Here, take A to be the projector on the state $\psi(0)$, i.e., $A = |\psi(0)\rangle\langle\psi(0)|$. We obtain $(\Delta A)^2 = \langle A\rangle(1-\langle A\rangle)$, with $\langle A\rangle = |\langle\psi(0)|\psi(t)\rangle|^2$. Inequality (2) then becomes more transparent by substituting $|\langle\psi(0)|\psi(t)\rangle|^2 =: \cos^2\phi(t)$, so that it take the form,

$$\left|\frac{d\phi}{dt}\right| \leqslant \frac{\Delta H}{\hbar},$$

also obtained by Vaidman. Integration yields

$$|\phi(t)| \leqslant \frac{\Delta H t}{\hbar},$$

which upon resubstitution gives Eq. (1). This inequality was also obtained by Fleming[4] and Bhattacharyya.[5] If we now take $|\psi(t)\rangle$ orthogonal to $|\psi(0)\rangle$, we obtain the limit for the question posed by Vaidman: in order to reach an orthogonal state, the system needs a time longer than $\pi\hbar/2\Delta H$. But we can read off more than that from inequality (1).

How long does it take until the nondecay probability $|\langle\psi(0)|\psi(t)\rangle|^2$ is $\frac{1}{2}$? (This time is usually called the half-life of $|\psi(0)\rangle$.) The inequality tells us that this half-life is greater than $\pi\hbar/4\Delta H$. Another interesting observation is that the inequality implies that for short times the nondecay probability must fall off more slowly than the parabola $1-(\Delta H t/\hbar)^2$ (at least as long as ΔH is finite). This result is the root of the well-known quantum Zeno paradox.[6] For yet another application, consider the "average lifetime"[4,5,7] defined by

$$\tau_{\text{ave}} = \int |\langle\psi(0)|\psi(t)\rangle|^2 dt.$$

By integrating inequality (1) one finds[4] $\tau_{\text{ave}}\Delta H \geqslant \pi/4$, which is only slightly less than the best possible bound $\tau_{\text{ave}}\Delta H \geqslant 3\pi 5^{1/2}/25$ obtained by Gislason, Sabelli and Wood.[7]

However, there are cases where the Mandelstam–Tamm inequality is not applicable. A notorious example is the Breit–Wigner state where

$$|\langle E|\psi(0)\rangle|^2 = \frac{\gamma}{\pi}\frac{1}{(E-E_0)^2+\gamma^2},$$

$$\langle\psi(0)|\psi(t)\rangle = e^{-(\gamma/2+iE_0)t/\hbar}.$$

The reason why the Mandelstam–Tamm inequality is inapplicable here is that ΔH is infinite, so that inequality (1) gives a trivial bound only for $t=0$. As a consequence, the nondecay rate for this state does not drop off quadratically for small times. However, this is not to say that there is no relationship between the uncertainty in energy and the evolution of this state. In fact this state is the textbook favorite example for illustrating the uncertainty relation between lifetime and energy.[8]

Indeed, the energy distribution for this state takes the shape of a simple peak, whose width depends on the value of γ. Therefore, one usually replaces the standard deviation ΔH in this case by a measure of uncertainty that is sensitive to the width of this peak. The customary choice is the width at half maximum. This gives $\delta E = \gamma$, which, in this example, happens to be inversely proportional to the lifetime of the state.

Thus it appears that ΔH can be a very unreasonable measure of uncertainty: it can be very large even when the distribution is very narrow. It is therefore desirable to have some inequality that does not rely on the standard deviation. In recent work[9] such an inequality has been derived. Let $|\psi(0)\rangle$ be any state and take $W_\alpha(E)$ to be the size of the shortest interval W such that

$$\int_W |\langle E|\psi(0)\rangle|^2 dE = \alpha.$$

Then $W_\alpha(E)$ gives a reasonable measure for the uncertainty in energy if α is less than but close to one (say $\alpha=0.9$). Note that $W_\alpha(E)$ is always finite. Further, let τ_β be the minimal time it takes for $|\psi(0)\rangle$ to evolve to a state $|\psi(\tau)\rangle$ such that

$$|\langle\psi(0)|\psi(\tau)\rangle|=\beta.$$

Then one can show

$$\tau_\beta W_\alpha(E)\geqslant 2\hbar\arccos\left(\frac{\beta+1-\alpha}{\alpha}\right),\quad\text{for }\beta\leqslant 2\alpha-1. \tag{3}$$

This result is actually easy to obtain by using the Mandelstam–Tamm inequality. The idea is simply to search for a decomposition of the initial state into two parts, one with large amplitude but limited decay rate, and another part which can decay rapidly, but with small amplitude. In fact, let P_W be the projector on the energy interval W mentioned above, and P_{W^c} be the projector on its complement. We can then write the state $|\psi(t)\rangle$ as

$$|\psi(t)\rangle=P_W|\psi(t)\rangle+P_{W^c}|\psi(t)\rangle$$
$$=\sqrt{\alpha}|\psi_W(t)\rangle+\sqrt{1-\alpha}|\psi_{W^c}(t)\rangle,$$

where ψ_W and ψ_{W^c} are normalized. Since the projectors P_W and P_{W^c} project onto disjoint intervals we have $P_WP_{W^c}=0$, and from this one can easily deduce

$$\langle\psi(0)|\psi(t)\rangle=\alpha\langle\psi_W(0)|\psi_W(t)\rangle$$
$$+(1-\alpha)\langle\psi_{W^c}(0)|\psi_{W^c}(t)\rangle. \tag{4}$$

For the first term on the right-hand side, we know from the Mandelstam–Tamm result that it cannot decay faster than $\alpha\cos(\Delta_W Ht)/\hbar$, where $\Delta_W H$ is now the standard deviation for the state ψ_W. And, by construction, the energy distribution of this state is contained in the interval W. Therefore, $\Delta_W H\leqslant W_\alpha(E)/2$. The second term in (4) may change much more rapidly, but this term will never become less than $-(1-\alpha)$. Thus we obtain:

$$|\langle\psi(0)|\psi(t)\rangle|\geqslant\alpha\cos\left(\frac{W_\alpha(E)t}{2\hbar}\right)-(1-\alpha),$$

which gives (3) above. It can also be shown that this inequality is sharp, i.e., the constant in the right-hand side is the best possible.

To illustrate this inequality, let us return to the Breit–Wigner state. Here, we have

$$W_\alpha(E)=2\gamma\tan\left(\frac{\alpha\pi}{2}\right),\quad \tau_\beta=-2\hbar\gamma^{-1}\log\beta.$$

For choices in the range $\beta\approx 0.3$–0.5, $\alpha\approx 0.7$–0.9 the product $W_\alpha(E)\tau_\beta$ is roughly ten times the theoretical minimum of inequality (3).

The main virtue of this inequality is that it shows that there is a general lower bound for the lifetime of all quantum states in terms of reasonable measures of the uncertainty in energy even if the standard deviation is infinite.

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[8] See, e.g., B. H. Bransden and C. J. Joachain, *Introduction to Quantum Mechanics* (Longman, London, 1989), pp. 73 and 511.

[9] J. Uffink and J. Hilgevoord, "Uncertainty principle and uncertainty relations," Found. Phys. **15**, 925–944 (1985), J. Hilgevoord and J. Uffink "The mathematical expression of the uncertainty principle," in *Microphysical Reality and Quantum Formalism*, edited by A. Van der Merwe (Kluwer, Dordrecht, 1988), pp. 91–114, and "A new view on the uncertainty principle," in *Sixty-two years of Uncertainty, Historical and Physical Inquiries into the Foundations of Quantum Mechanics*, edited by A. I. Miller (Plenum, N.Y., 1990), pp. 121–139.

SUBTLE IS THE LORD

One of Einstein's famous quotations, which you can even see in the old Fine Hall above the mantelpiece, reads in German, "Raffiniert ist der Herr Gott aber boshaft ist Er nicht," of which an English translation may read as follows: "God is subtle, but He isn't malicious." And the interpretation usually was: "It might be difficult to find the laws of nature, but it is not impossible." There is also a somewhat different interpretation, because once Einstein said to us, "I have had second thoughts. Maybe God is malicious after all." But what he meant was something very specific. It was that God makes us believe that we understand something, when in reality we are very far from it. And Einstein was very much concerned that one should not be uncritical enough to be misled in this way.

Valentine Bargmann, in *Some Strangeness in the Proportion*, edited by Harry Woolf (Addison–Wesley, Reading, MA, 1980), pp. 480–481.